Two Partitions of Unity

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Halmos 1969: Two projections $P, Q \in B(\mathcal{H})$ in generic position are of the form

$$P = \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix},$$

with contractions $C, S \in B(\mathcal{K})$ satisfying $C^2 + S^2 = I_{\mathcal{K}}$. (Two Projection Theorem)

Pedersen 1968: The universal C*-algebra generated by two projections p, q is

$$\{f \in C([0,1], M_2) \mid f(0), f(1) \text{ diagonal}\}.$$

For a historical overview see Böttcher, Spitkovsky: A gentle guide to the basics of two projections theory (2009).

This talk: A class of C*-algebras generated by two partitions of unity + a generalization of Halmos'/Pedersen's results to more projections.

Why you can't generalize Halmos' Two Projections Theorem

Theorem (Davis 1955)

For any Hilbert space $\mathcal{H}$ there exist three projections $P_1, P_2, P_3 \in B(\mathcal{H})$ such that $B(\mathcal{H})$ is generated by I, P_1, P_2, P_3 as a von Neumann algebra.

... and when you can:

- Banach algebra generated by two idempotents (Roch–Silbermann 1988)
- C*-/Banach algebra generated by one projection/idempotent and one partition of unity with additional relations (Böttcher et al. 1996)

Definition

Given a bipartite graph $G = (U, V, E)$ let $C^*(G)$ be the universal C^*-algebra generated by a family of projections $(p_x)_{x \in U \cup V}$ subject to the following relations:

$$\sum_{u \in U} p_u = 1 = \sum_{v \in V} p_v, \tag{GP1}$$

$$p_u p_v = 0 \text{ if } \{u, v\} \notin E. \tag{GP2}$$

Example

$C^*(K_{m,n}) = \mathbb{C}^m *_{\mathbb{C}} \mathbb{C}^n$, $C^*(K_{2,2}) = C^*(p, q)$ from Pedersen's theorem.

There is a connection to Trieb–Weber–Zenner's *hypergraph C*-algebras*:

Theorem (S. 2025)

To know which hypergraph C-algebras are nuclear you need to know which bipartite graph C*-algebras are nuclear.*

Proposition (S. 2026+)

Let $G = (U, V, E)$ be a bipartite graph. Then $C^*(G)$ is the universal C^*-algebra generated by a family of elements $(x_e)_{e\in E}$ satisfying

$$x_e^* x_f = 0 \qquad \text{if } e \cap f \cap U = \emptyset, \tag{GC1}$$

$$x_e x_f^* = 0 \qquad \text{if } e \cap f \cap V = \emptyset, \tag{GC2}$$

$$\left(\sum_{e\in E} x_e^*\right) x_f = x_f, \tag{GC3}$$

$$x_e \left(\sum_{f\in E} x_f^*\right) = x_e, \tag{GC4}$$

for all edges $e, f \in E$, respectively. In particular, the x_e are contractions which satisfy $x_e x_e^* x_e = x_e^2$.

Question: When is $C^*(G) \cong C^*(H)$ for two bipartite graphs G and H?

Lemma

The one-dimensional irreducible representations of $C^*(G)$ are in one-to-one correspondence with the edges of G.

How to see this: Let $\pi : C^*(G) \to \mathbb{C}$ be a *-homomorphism. The $\pi(p_v)$ for $v \in V$ are pairwise orthogonal projections, thus there is exactly one $v_0 \in V$ with $0 \neq \pi(p_{v_0}) = 1$.

Lemma

The two-dimensional irreducible representations of $C^*(G)$ are in one-to-one correspondence with subgraphs of G that are isomorphic to $K_{2,2}$.

Theorem (S. 2026+)

It is

$$C^*(G) \cong C^*(H) \quad \Leftrightarrow \quad \mathrm{Spec}_{\leq 2}(C^*(G)) \cong \mathrm{Spec}_{\leq 2}(C^*(H)),$$

where $\mathrm{Spec}_{\leq 2} \subset \mathrm{Spec}$ is the space of one- and two-dimensional irreducible representations.

Generalizing Halmos'/Pedersen's theorem to bipartite graph C*-algebras

Main problem for bipartite graph C*-algebras

Let $G = (U, V, E)$ be a bipartite graph. We want to find a concrete description

$$C^*(G) \cong \{f \in C(X, M_n) \mid \quad (?) \quad \}$$

for some space X, $n \in \mathbb{N}$ and conditions "(?)" generalizing

$$C^*(K_{2,2}) = C^*(p, q) \cong \{f \in C([0,1], M_2) \mid f(0), f(1) \text{ diagonal}\}.$$

In particular, this should help answer the following question:

Question: For which bipartite graphs G is $C^*(G)$ nuclear?

Proposition

If $K_{2,3} \subset G$, then $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$ is a quotient of $C^*(G)$ and the latter is not nuclear.

Dream: Find a concrete description as above for all G with $K_{2,3} \not\subset G$ and thus prove that these algebras are nuclear.

Reality: We can show that for the hypercubes Q_n.

Related work on N projections

Böttcher–Gohberg–Karlovich–Krupnik–Roch–Silbermann–Spitkovsky (1996) investigated Banach algebras $\mathcal{B}$ generated by one idempotent P and a partition of unity $p_1, \dots, p_{2N}$ satisfying

$$P(p_{2i-1} + p_{2i})P = (p_{2i-1} + p_{2i})P$$

and

$$(1 - P)(p_{2i} + p_{2i+1})(1 - P) = (p_{2i} + p_{2i+1})(1 - P)$$

for all i.

These Banach algebras occur as algebras generated by singular integral operators.

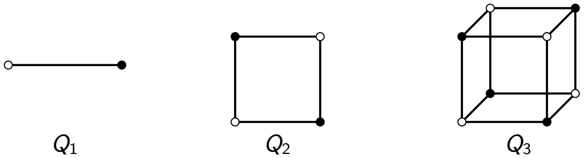
Theorem (Böttcher et al. "N Projections Theorem" 1996)

For every $x \in \sigma(\mathcal{B}) \setminus \{0, 1\}$ one has a $*$-homomorphism

$$F_x : \mathcal{B} \to M_{2N}$$

with $F_x(p_i) = \mathrm{diag}(\dots, 0, 1, 0, \dots)$. Further, ...

Hypercubes

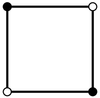


The hypercube Q_n has 2^n vertices. The vertices are $\{1, \ldots, 2^n\}$ and two vertices are connected if they differ in exactly one binary digit. It is a bipartite graph with bipartition

$$U_n = \{\text{vertices with an even number of 1's in the binary representation}\},$$
$$V_n = \{\text{vertices with an odd number of 1's in the binary representation}\}.$$

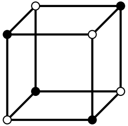
Example

1. For $n = 1$ it is $C^*(Q_1) = \mathbb{C}$.
2. For $n = 2$ one has $Q_2 = K_{2,2}$ and thus



$$C^*(Q_2) = C^*(p, q) \cong \{f \in C([0,1], M_2) \mid f(0), f(1) \text{ diagonal}\}.$$

3. For $n = 3$, $C^*(Q_3)$ is generated by $4 + 4$ projections p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4 satisfying



$$p_1 + p_2 + p_3 + p_4 = 1 = q_1 + q_2 + q_3 + q_4,$$
$$p_i q_j = 0 \quad \text{if } i = j.$$

Goal: Describe $C^*(Q_n)$ as $\{f \in C(X, M_k) \mid (?)\}$.

Lemma

For every vertex x, the corner $p_x C^*(Q_n) p_x$ is commutative.

Proof.

Combinatorial argument: The corner has dense subset spanned by $p_x p_{y_1} \dots p_{y_k} p_x$ for paths $x y_1 \dots y_k x$ in the hypercube Q_n. For a segemnt $y_i y_{i+1} i_{i+2}$ with $y_i \neq y_{i+2}$ there is a unique $y'_{i+1} \neq y_{i+1}$ such that $y_i y'_{i+1} y_{i+2}$ is also a path. The relations (GP1) and (GP2) yield

$$p_{y_i} p_{y_{i+1}} p_{y_{i+2}} = -p_{y_i} p_{y'_{i+1}} p_{y_{i+2}}.$$

Using this one shows

$$p_x p_{y_1} \dots p_{y_k} p_x = p_x p_{y_k} \dots p_{y_1} p_x = (p_x p_{y_1} \dots p_{y_k} p_x)^*.$$

□

Hypercube C*-algebras (3)

Recall that we want to generalize

$$C^*(p,q) \cong C^*(Q_2) \cong \{f \in C([0,1], M_2) : f(0), f(1) \text{ are diagonal}\}.$$

Recall the n-simplex $\Delta_n = \{[t_0, \dots, t_n] \in [0,1]^{n+1} : t_0 + \cdots + t_n = 1\}$. A point $\mathbf{t} \in \Delta_n$ is in the boundary of Δ_n if at least one of its entries is zero.

For every boundary point $\mathbf{t} \in \partial\Delta_n$ and matrix $A \in M_{2^{n-1}}$ we say that A is in **t-block diagonal form** if it can be written as a block matrix where the position of the blocks depends on the position of $\mathbf{t}$ on the boundary of Δ_n.

Theorem (S. 2026+)

There is an isomorphism

$$C^*(Q_n) \cong \{f \in C(\Delta_{n-1}, M_{2^{n-1}}) : f(\mathbf{t}) \text{ is in } \mathbf{t}\text{-block diagonal form}\}.$$

Application to magic isometries

A 2×4-magic isometry is a 2×4-matrix with entries from a C*-algebra A such that the rows form partitions of unity and the projections are orthogonal along columns.

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$

The universal C*-algebra generated by its entries is $C^*(Q_3)$.

Problem (Banica–Skalski–Sołtan 2012): Can you complete any such magic isometry to a 4×4-magic unitary by adding additional projections in a third and fourth row?

Proposition

Any 2×4*-magic isometry can be completed to a* 4×4*-magic unitary.*

Proof.

We have an explicit description of the universal C*-algebra generated by the entries of the magic isometry. On the other hand, Banica–Collins (2008) have proved a faithful representation of $C(S_4^+)$, the universal C*-algebra generated by the entries of a 4×4-magic unitary. One can show that every irrep of $C^*(Q_3)$ is a restriction of an irrep of $C(S_4^+)$, and the claim follows. □

Thank you!

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