Hypergraph C*-algebras

Björn Schäfer

Saarland University

20th Workshop: Noncommutative Probability, Operator Algebras and Related Topics, with Applications at Będlewo 2025

Introduction

Moritz Weber, Mirjam Trieb and Dean Zenner introduced hypergraph C*-algebras as a generalization of graph C*-algebras.

Unlike graph C*-algebras, these can be non-nuclear.

To study nuclearity of hypergraph C*-algebras, one needs to study certain C*-algebras $C^*(G)$ generated by two partitions of unity associated to a bipartite graph G.

For the hypercubes Q_n, $C^*(Q_n)$ yields a nice generalization of $C^*(p, q)$, the universal C*-algebra generated by two projections p and q.

Outline

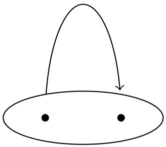
1. Hypergraph C*-algebras
2. Bipartite graph C*-algebras
   - Nuclearity
   - Classification
   - Hypercubes

Hypergraph C^*-Algebras (1/3)

A (finite) hypergraph $\mathrm{H}\Gamma$ is a tuple (E^0, E^1, r, s), where

- E^0 is the (finite) set of *vertices* of $\mathrm{H}\Gamma$,
- E^1 is the (finite) set of *edges* of $\mathrm{H}\Gamma$,
- $r : E^1 \to \mathcal{P}(E^0)$ maps every edge to its *range* set,
- $s : E^1 \to \mathcal{P}(E^0) \setminus \{\emptyset\}$ maps every edge to its *source* set.

Example (1)



$$E^0 = \{v, w\},$$

$$E^1 = \{e\},$$

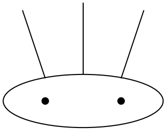
$$s(e) = r(e) = \{v, w\}.$$

Hypergraph C^*-Algebras (1/3)

A (finite) hypergraph $\mathrm{H}\Gamma$ is a tuple (E^0, E^1, r, s), where

- E^0 is the (finite) set of *vertices* of $\mathrm{H}\Gamma$,
- E^1 is the (finite) set of *edges* of $\mathrm{H}\Gamma$,
- $r : E^1 \to \mathcal{P}(E^0)$ maps every edge to its *range* set,
- $s : E^1 \to \mathcal{P}(E^0) \setminus \{\emptyset\}$ maps every edge to its *source* set.

Example (2)



$E^0 = \{v, w\}$,

$E^1 = \{e_1, e_2, e_3\}$,

$s(e_i) = \{v, w\}$ for all $i \leq 3$,

$r(e_i) = \emptyset$ for all $i \leq 3$.

$\mathrm{H}\Gamma$ is called **undirected** if every edge has empty range.

Hypergraph C^*-Algebras (2/3)

Let $\mathrm{H}\Gamma = (E^0, E^1, r, s)$ be a (finite) hypergraph.

Definition (Weber, Zenner '22)

The hypergraph C^*-algebra $C^*(\mathrm{H}\Gamma)$ is the universal C^*-algebra generated by

- pairwise orthogonal projections p_v for $v \in E^0$,
- partial isometries s_e for $e \in E^1$,

satisfying three relations (HR1), (HR2), (HR3).

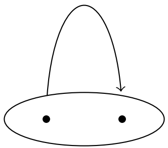
$$s_e^* s_f = \begin{cases} \delta_{ef} \sum_{v \in r(e)} p_v, & r(e) \neq \emptyset, \\ \delta_{ef} s_e, & \text{otherwise}, \end{cases} \quad \text{for all } e, f \in E^1, \tag{HR1}$$

$$s_e s_e^* \leq \sum_{v \in s(e)} p_v \text{ for all } e \in E^1, \tag{HR2}$$

$$p_v \leq \sum_{e : v \in s(e)} s_e s_e^* \text{ for all } v \in E^0 \text{ with } \exists e \in E^1 : v \in s(e). \tag{HR3}$$

Hypergraph C*-Algebras (3/3)

Recall the hypergraph $\mathrm{H}\Gamma$ sketched below.



Then $C^*(\mathrm{H}\Gamma) = \mathbb{C}^2 *_{\mathbb{C}} C(S^1)$ is **not nuclear**. [Zenner 2021]

Question: Which hypergraph C*-algebras are nuclear?

Theorem (S. 2024)

For every hypergraph $\mathrm{H}\Gamma$ one can construct an undirected hypergraph $\mathrm{H}\Delta$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if and only if $C^*(\mathrm{H}\Delta)$ is nuclear.

Bipartite Graph C^*-Algebras

Definition

For a bipartite graph $G = (U, V, E)$ with vertex sets U and V let $C^*(G)$ be the universal C^*-algebra generated by two families of projections $(p_u)_{u\in U}$ and $(p_v)_{v\in V}$ such that

- $\sum_{u\in U} p_u = 1 = \sum_{v\in V} p_v$,
- $p_u p_v = 0$ if $\{u, v\} \notin E$.

These C^*-algebras arise as hypergraph C*-algebras of undirected hypergraphs and as subalgebras of hypergraph C*-algebras.

Proposition

A dense subset of $C^*(G)$ is spanned by products $p_{x_1} \dots p_{x_k}$ where $x_1 \dots x_k$ is a path in G.

Example

If $G = K_{n,m}$ then $C^*(G) \cong \mathbb{C}^n *_{\mathbb{C}} \mathbb{C}^m \cong C^*(\mathbb{Z}_n * \mathbb{Z}_m)$.

Nucleárity

Question: When is $C^*(G)$ nuclear?

Example

$C^*(K_{2,3}) \cong \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$ is not nuclear,
$C^*(K_{2,2}) \cong \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2 \cong C^*(p, q)$ is nuclear and subhomogeneous.

Proposition

If $G \subset H$, then $C^*(G)$ is a quotient of $C^*(H)$.

Thus, $C^*(G)$ can only be nuclear if $K_{2,3} \not\subset G$.

Open Question: Is $C^*(G)$ nuclear if and only if $K_{2,3} \not\subset G$?

We will later see that for the hypercubes Q_n one has $K_{2,3} \not\subset Q_n$ and $C^*(Q_n)$ is nuclear.

Classification

Question: When is $C^*(G) \cong C^*(H)$ for two bipartite graphs G and H?

Observation

There is a correspondence between

- *1-dimensional irreducible representations of $C^*(G)$ and edges of G,*
- *2-dimensional irred. rep. of $C^*(G)$ and subgraphs $G \supset G' \cong K_{2,2}$.*

Let $\mathrm{Spec}_{\leq 2}(C^*(G)) \subset \mathrm{Spec}(C^*(G))$ be the subspace consisting of (equivalence classes of) 1- and 2-dimensional irreducible representations of $C^*(G)$.

Theorem

We have $C^*(G) \cong C^*(H) \Leftrightarrow \mathrm{Spec}_{\leq 2}(C^*(G)) \cong \mathrm{Spec}_{\leq 2}(C^*(H))$.

Hypercube C*-Algebras

Let Q_n be the n-dimensional hypercube seen as a bipartite graph.

Recall

$$C^*(p,q) \cong C^*(Q_2) \cong \{f \in C([0,1], M_2) : f(0), f(1) \text{ are diagonal}\}.$$

Recall the n-simplex $\Delta_n = \{[t_0, \dots, t_n] \in [0,1]^{n+1} : t_0 + \cdots + t_n = 1\}$. A point $\mathbf{t} \in \Delta_n$ is in the boundary of Δ_n if at least one of its entries is zero.

For every boundary point $\mathbf{t} \in \partial\Delta_n$ and matrix $A \in M_{2^{n-1}}$ we say that A is in $\mathbf{t}$-*block diagonal form* if it can be written as a block matrix where the position of the blocks depends on the position of $\mathbf{t}$ on the boundary of Δ_n.

Theorem

$C^*(Q_n) \cong \{f \in C(\Delta_{n-1}, M_{2^{n-1}}) : f(\mathbf{t})$ *is in* $\mathbf{t}$*-block diagonal form*$\}$.

Thank you!